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B.M.S. COLLEGE FOR WOMEN

BENGALURU -560004

I SEMESTER END EXAMINATION – APRIL - 2024

M.Sc. MATHEMATICS - REAL ANALYSIS
(CBCS Scheme – F+R)

Course Code: MM102T — **QP Code: 11002**
Duration: 3 Hours — **Max. Marks: 70**

Instructions: ***1) All questions carry equal marks.***
2) Answer any five full questions.

1. a) Show that $x^2 \in R[x^2]$ on $[0,1]$.
   b) If P^* is a refinement of a partition P of $[a,b]$, then show that
   $$L(P,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P,f,\alpha)$$
   c) Prove that if $f \in R[\alpha]$, then $-f \in R[\alpha]$ on $[a,b]$.

(5+5+4)

2. a) Prove that if $f \in R[\alpha]$ on $[a,b]$, then $|f| \in R[\alpha]$ on $[a,b]$. Also give an example of a function f such that $|f| \in R$ on $[0,1]$ and $f \notin R$ on $[0,1]$.
   b) Prove that if f is monotonic and α is continuous on $[a,b]$, then $f \in R[\alpha]$ on $[a,b]$.
   c) If $f \in R[\alpha]$ on $[a,b]$, $f \in [m,M]$ and ϕ is a continuous function on $[m,M]$, then prove that $\phi\big(f(x)\big) \in R[\alpha]$ on $[a,b]$.

(4+5+5)

3. a) If f is continuous on $[a,b]$ and α is monotonic on $[a,b]$, then prove that
   $$\int_a^b f\,d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(\xi)[f(b) - f(a)],$$
   where $\xi \in (a,b)$.
   b) Show that a function of bounded variation on $[a,b]$ is bounded.
   c) Calculate the total variation function of $f(x) = x - [x]$ on $[0,2]$, where $[x]$ is the greatest integer not exceeding x.

(5+4+5)

4. a) Prove that a series of functions $\sum f_n$ defined on $[a,b]$ converges uniformly on $[a,b]$ if and only if for every $\epsilon > 0$ and for all $x \in [a,b]$, there exists an integer N such that
   $$\left|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)\right| < \epsilon, \quad \forall\, n \geq N, \quad p \geq 1.$$
   b) If $\{f_n(x)\}$ is a sequence of functions converges to $f(x)$ defined on $[a,b]$ and $M_n = \underset{x\in[a,b]}{Sup} |f_n(x) - f(x)|$, then prove that $\{f_n(x)\}$ converges uniformly to $f(x)$ on $[a,b]$ if and only if $M_n \to 0$ as $n \to \infty$.

c) Show that $\{nx\, e^{-nx^2}\}$ is not uniformly convergent on $[0,1]$.

(5+5+4)

5. a) Let $\{f_n(x)\}$ be a sequence of differentiable functions on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some x_0 on $[a,b]$. If the sequence $\{f_n'(x)\}$ converges uniformly on $[a,b]$, then show that $\{f_n(x)\}$ converges uniformly to a function $f(x)$ on $[a,b]$ and $\lim_{n\to\infty} f_n'(x) = f'(x)$.

   b) Suppose $f_n \to f$ uniformly on $[a,b]$ and if $x_0 \in [a,b]$ such that $\lim_{x\to x_0} f_n(x) = a_n$ for $n = 1, 2, 3, \cdots$. Then prove that

   (i) $\{a_n\}$ converges

   (ii) $\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to x_0} f_n(x)$.

(7+7)

6. a) Define a k −cell in $\mathbb{R}^k$. Prove that every k −cell is compact.

   b) Prove that every infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

(7+7)

7. a) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and if f is differentiable at $x \in E$. Then prove that the partial derivatives $D_j f_i(x)$exist and $f'(x)e_j = \sum_{i=1}^{m} D_j f_i(x) u_i$ $(1 \le j \le n)$.

   b) If $\phi: X \to X$ is a contraction mapping on a complete metric space X then prove that ϕ has a unique fixed point.

   c) If $T_1, T_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$ then prove that

   (i) $\|T_1 + T_2\| \le \|T_1\| + \|T_2\|$

   (ii) $\|\alpha T_1\| = |\alpha| \|T_1\|$

(4+6+4)

8. State and prove the implicit function theorem. **(14)**

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